Maxwell's equations [macroscopic]

$$\vec{\nabla}\times\tilde{\vec{E}}(\vec{r},t)+\frac{\partial}{\partial t}\tilde{\vec{B}}(\vec{r},t) = 0 \qquad \vec{\nabla}\times\tilde{\vec{H}}(\vec{r},t)-\frac{\partial}{\partial t}\tilde{\vec{D}}(\vec{r},t) = \tilde{\vec{\jmath}}_{\text{macr}}(\vec{r},t)$$
$$\vec{\nabla}\,\tilde{\vec{D}}(\vec{r},t) = \tilde{\rho}_{\text{ext}}(\vec{r},t) \qquad \vec{\nabla}\,\tilde{\vec{B}}(\vec{r},t) = 0$$

For monochromatic [$f(\vec{r},t)=f'(\vec{r})\,\mathrm{e}^{-\mathrm{i}\,\omega t}$] fields in homogeneous, isotropic, linear, non-magnetic media it is:
$\breve{\vec{D}}=\varepsilon_0\breve{\varepsilon}_r\breve{\vec{E}}$, $\breve{\varepsilon}_r=\varepsilon_r+\mathrm{i}\,\varepsilon_r'$, $\varepsilon_r,\,\varepsilon_r'\in\mathbb{R}$, $\breve{\vec{B}}=\mu_0\breve{\vec{H}}$.

It follows: $-\Delta\vec{E}(\vec{r})=k_0^2\breve{\varepsilon}_r\vec{E}(\vec{r})$, $-\Delta\vec{H}(\vec{r})=k_0^2\breve{\varepsilon}_r\vec{H}(\vec{r})$ - Helmholtz-equations , with $k_0=\frac{2\pi}{\lambda_0}$.

One possible set of solutions are the plane waves $\vec{E}(\vec{r})=\breve{\vec{E}}\,\mathrm{e}^{\mathrm{i}\,\breve{\vec{k}}\vec{r}}$, $\vec{H}(\vec{r})=\breve{\vec{H}}\,\mathrm{e}^{\mathrm{i}\,\breve{\vec{k}}\vec{r}}$
, with $\breve{\vec{k}}=\vec{k}+\mathrm{i}\,\vec{k}'$, $\vec{k},\,\vec{k}'\in\mathbb{R}^3$.

Using Helmholtz this yields: $\breve{\vec{k}}^2=k_0^2\,\breve{\varepsilon}_r$, with $\breve{\varepsilon}_r=\breve{n}^2$.

All field-components can be determined from $\breve{E}_x$ and $\breve{E}_y$: $\breve{E}_z=-\frac{1}{\breve{k}_z}[\breve{k}_x\breve{E}_x+\breve{k}_y\breve{E}_y]$, $\breve{\vec{H}}=\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{\breve{\vec{k}}\times\breve{\vec{E}}}{k_0}$.

$$\vec{E}(\vec{r})=\underbrace{\hat{\vec{E}}(\vec{r})}_{\text{diffractive field factor}}\overbrace{\mathrm{e}^{\mathrm{i}\,\breve{\psi}(\vec{r})}}^{\text{geometric field factor}}\ ,\quad \breve{\psi}(\vec{r})=\psi(\vec{r})+\mathrm{i}\,\psi'(\vec{r})\ ,\ \psi,\,\psi'\in\mathbb{R}\quad\text{and}\quad \hat{E}_j(\vec{r})=\left|\hat{E}_j(\vec{r})\right|\mathrm{e}^{\mathrm{i}\,\varphi_j(\vec{r})}\ ,\ \varphi_j(\vec{r})\in\mathbb{R}\ ,\ j\in\{1,2,3\}.$$

Each set of connected points with $\psi(\vec{r})=const.$ makes up a wavefront [wavefront factor $\mathrm{e}^{\mathrm{i}\,\psi(\vec{r})}$].

For **geometric optics** the geometric field factor should be the dominant component in the Maxwell equations:

$$\frac{\partial\hat{E}_j(\vec{r})}{\partial x_k}\ll\mathrm{i}\,\hat{E}_j(\vec{r})\frac{\partial\breve{\psi}(\vec{r})}{\partial x_k}$$

Maxwell:

$$[\vec{\nabla}\tilde{\Psi}]\times\tilde{\vec{E}}(\vec{r},t)-\mathrm{i}\,\omega\mu_0\tilde{\vec{H}}(\vec{r},t) = 0 \qquad [\vec{\nabla}\tilde{\Psi}]\times\tilde{\vec{H}}(\vec{r},t)+\mathrm{i}\,\omega\varepsilon_0\varepsilon_r\tilde{\vec{E}}(\vec{r},t) = 0$$
$$[\vec{\nabla}\tilde{\Psi}]\,\tilde{\vec{E}}(\vec{r},t) = 0 \qquad [\vec{\nabla}\tilde{\Psi}]\,\tilde{\vec{H}}(\vec{r},t) = 0$$

Then it follows $[\vec{\nabla}\breve{\psi}(\vec{r})]^2-k_0^2\breve{n}^2=0$ - Eikonal equation.

The Eikonal is usually defined as $s=\frac{1}{k_0}\psi(\vec{r})$. [$\breve{\vec{k}}=k_0\breve{n}\vec{s}=\vec{\nabla}\breve{\psi}$]

Diffractive effects govern propagation in region of focus. In the far-field [or for very short propagation distances] a geometric description suffices.

A measure for the magnitude of the importance of diffraction is the Fresnel number $F(z)$: $\begin{cases}\ll 1 & \text{, far-field}\\ \sim 1 & \text{, something in between}\\ \gg 1 & \text{, near-field}\end{cases}$.

Depending on the definition of the radius of the beam [„field tube" is beam inside of radius], the absolute values of the Fresnel-number can differ; their relative dependency on z however remains.

For a rect-function of diameter x_0 at the beam waist and a beam-radius extending to the first lateral root of the E-field, it yields $F(z)=\frac{x_0^2 n}{\lambda_0\,|z|}=\frac{\tilde{z}}{|z|}$. [$\tilde{z}$ corresponds to the Rayleigh length of a Gaussian.]

Geometrical optics: mainly the evolution of the wavefront governs the spatial distribution of the wave.

Diffractive optics: mainly the amplitude evolution governs the spatial distribution of the wave.

E.g. for an undamped par-axial Gaussian:

$$\vec{E}(\vec{r}) = \tilde{\vec{E}}(\vec{r})\,\mathrm{e}^{\mathrm{i}\,\tilde{\Psi}(\vec{r})} \quad , \quad \text{with} \quad \tilde{E}_{x_i}(\vec{\rho},z) = E_{0,x_i}\frac{w_0}{w(z)}\,\mathrm{e}^{-\frac{\vec{\rho}^2}{w^2(z)}} \quad \text{and} \quad \tilde{\Psi}(\vec{\rho},z) = k_0 n z - \Phi(z) + \frac{k_0 n}{2R(z)}\vec{\rho}^2 \quad ;$$

here it is $z_R = \frac{\pi n}{\lambda_0}{w_0}^2$ - Rayleigh length, $\Phi(z) = \arctan\left(\frac{z}{z_R}\right)$ - Gouy phase shift, $w(z) = w_0\sqrt{1+\left[\frac{z}{z_R}\right]^2}$ - beam radius, $R(z) = z + \frac{{z_R}^2}{z}$ - phase profile radius of curvature.

Diffractive effects:

From the divergence of the amplitude profile of the Gaussian beam one reads the following angular effect in the far-field:

$$\tilde{\theta}' = \arctan\left(\frac{w(z)}{z}\right) = \arctan\left(\frac{w_0\sqrt{1+\left[\frac{z}{z_R}\right]^2}}{z}\right) \overset{z\gg z_R}{\approx} \arctan\left(\frac{w_0}{z_R}\right) = \arctan\left(\frac{\lambda}{\pi n w_0}\right) \quad ;$$

i.e. the effect of the diffractive field factor in the beam-waist leads to an angular effect in the far-field as seen here. If one now considers each transversal profile of the beam $\tilde{\vec{E}}(\vec{\rho},z)$ at each z as a beam-waist of a separate beam [as one neglects the phase-profile ⇒ flat phase profile [which exists physically only in the beam-waist]], then the purely diffractive angular effect in the far-field reads as follows:

$$\tilde{\theta}'' = \arctan\left(\frac{\lambda}{\pi n w(z)}\right) = \arctan\left(\frac{\lambda}{\pi n w_0\sqrt{1+\left[\frac{z}{z_R}\right]^2}}\right)$$

Geometric effects:

By looking at the curvature of the phase-profile one finds:

$$\theta_{\tilde{\Psi}}(z) = \arcsin\left(\frac{w(z)}{R(z)}\right) = \arcsin\left(\frac{w_0 z}{{z_R}^2\sqrt{1+\left[\frac{z}{z_R}\right]^2}}\right)$$

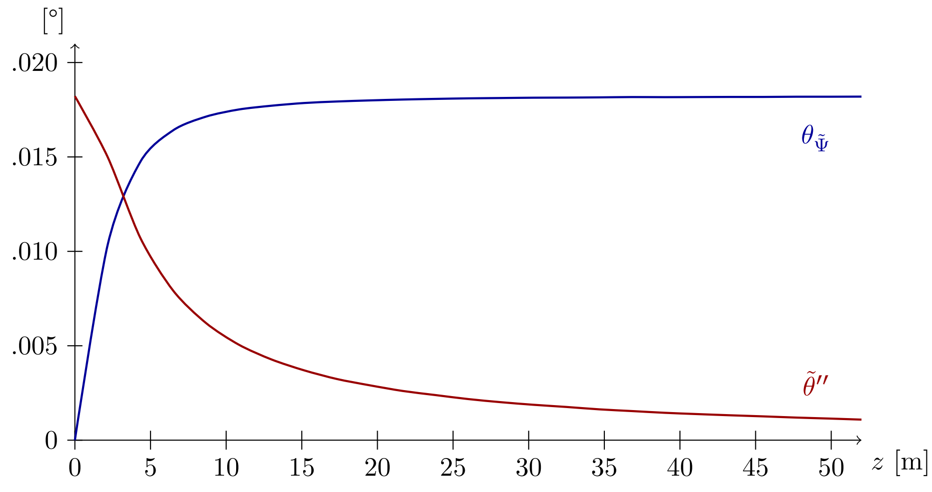


Abbildung 1: Comparison of diffractive and geometrical optic effects over the distance of the beam-waist for a Gaussian beam.

One sees, that the diffractive effects govern for small distances, however become negligible after just a few z_R.

More than geometric optics

small [spectrum of plane waves [SPW]]

Field tube: region in $\mathbb{R}^3$ where the envelope of the amplitude is bigger than a certain value.

The propagation direction be z [transversal plane (x,y) is called „eigencoordinate system“] and the field have only one focus [where the transversal profile of the field tube is smallest, z_{focus}]. The region $[z_{\text{focus}} - \Delta z, z_{\text{focus}} + \Delta z]$, with Δz not too big, is called focal region.

Note that different field components may have different field tubes, directions, foci, ... In linear Optics every field however can be described by a superposition of different fields, i.e. by different field tubes.

[Maxwell's equations are solvable directly via numerical methods - this however is numerically very demanding [i.e. takes a long time and needs a lot of computation power].]

Alternatively one can assume the field to be decomposable in a „spectrum of plane waves“ [SPW], propagate the plane waves analytically through a homogenous medium and then find the new field by superposition of the propagated fields: In assuming homogeneous, isotropic, non-magnetic, free-charge- and free-current-free, optical media $[\check{\varepsilon}_r(\omega) = \varepsilon_r(\omega) + \mathrm{i}\,\frac{\sigma(\omega)}{\omega\varepsilon_0},\ \check{n}(\omega) = \sqrt{\check{\varepsilon}_r(\omega)} = n(\omega) + \mathrm{i}\,n'(\omega)]$, shortening the notation by introducing $\vec{V} := (E_x, E_y, E_z, H_x, H_y, H_z)$, employing the Fourier transformations [$\hat{V}_l(\vec{\kappa}, z) = \mathrm{FT}\left(V_l(\vec{\rho}, z)\right) = \frac{1}{2\pi}\iint_{\mathbb{R}^2} V_l(\vec{\rho}, z)\,\mathrm{e}^{-\mathrm{i}\,\vec{\kappa}\vec{\rho}}\,\mathrm{d}^2\rho$, $V_l(\vec{\rho}, z) = \mathrm{FT}^{-1}\left(\hat{V}_l(\vec{\kappa}, z)\right) = \frac{1}{2\pi}\iint_{\mathbb{R}^2} \hat{V}_l(\vec{\kappa}, z)\,\mathrm{e}^{+\mathrm{i}\,\vec{\kappa}\vec{\rho}}\,\mathrm{d}^2\kappa$] and the „angular spectrum“ $\hat{V}_l(\vec{\kappa}) = \hat{V}_l(\vec{\kappa}, z)\,\mathrm{e}^{-\mathrm{i}\,\check{\check{k}}_z z}$ with $\check{k}_z = \sqrt{{k_0}^2 \check{n}^2(\omega) - \vec{\kappa}2}$, $\vec{\kappa} = (k_x, k_y, 0) \in \mathbb{R}^3$ and $\vec{\rho} = (x, y, 0)$, it follows from the angular spectrum that

$$\hat{V}_l(\vec{\kappa}, z^{\text{out}}) = \hat{V}_l(\vec{\kappa}, z^{\text{in}})\,\mathrm{e}^{\mathrm{i}\,\check{k}_z[z^{\text{out}} - z^{\text{in}}]} = \hat{V}_l(\vec{\kappa}, z^{\text{in}})\,\mathrm{e}^{\mathrm{i}\,\check{k}_z \Delta z}\ .$$

By using the Fourier transformations this can be formulated with the real-space field values:

$$V_l(\vec{\rho}, z^{\text{out}}) = \mathrm{FT}^{-1}\left(\mathrm{FT}\left(V_l(\vec{\rho}, z^{\text{in}})\right)\mathrm{e}^{\mathrm{i}\,\check{k}_z \Delta z}\right)\ .$$

Because for plane waves in homogeneous space the following equations can be derived from Maxwell's equations, numerically only E_x and E_y will have to be evaluated.

$$E_z(\vec{\rho}, z) = \mathrm{FT}^{-1}\left(-\frac{1}{\check{k}_z}[k_x\,\mathrm{FT}\left(E_x(\vec{\rho}, z)\right) + k_y\,\mathrm{FT}\left(E_y(\vec{\rho}, z)\right)]\right)\ ,$$

$$H_x(\vec{\rho}, z) = -\frac{1}{k_0}\sqrt{\frac{\varepsilon_0}{\mu_0}}\,\mathrm{FT}^{-1}\left(\frac{1}{\check{k}_z}\left[k_x k_y\,\mathrm{FT}\left(E_x(\vec{\rho}, z)\right) + [{k_y}^2 + \check{k}_z^2]\,\mathrm{FT}\left(E_y(\vec{\rho}, z)\right)\right]\right)\ ,$$

$$H_y(\vec{\rho}, z) = \frac{1}{k_0}\sqrt{\frac{\varepsilon_0}{\mu_0}}\,\mathrm{FT}^{-1}\left(\frac{1}{\check{k}_z}\left[[{k_x}^2 + \check{k}_z^2]\,\mathrm{FT}\left(E_x(\vec{\rho}, z)\right) + k_x k_y\,\mathrm{FT}\left(E_y(\vec{\rho}, z)\right)\right]\right)\ ,$$

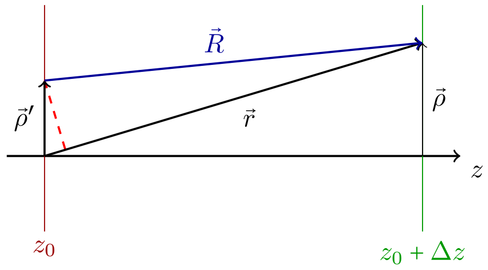
$$H_z(\vec{\rho}, z) = -\frac{1}{k_0}\sqrt{\frac{\varepsilon_0}{\mu_0}}\,\mathrm{FT}^{-1}\left(k_y\,\mathrm{FT}\left(E_x(\vec{\rho}, z)\right) - k_x\,\mathrm{FT}\left(E_y(\vec{\rho}, z)\right)\right)\ .$$

Numerical challenges are:

- Damped waves can vanish into numerical noise [i.e. can't be recovered by backward propagation - and vice versa].
- $\mathrm{FT}\left(V_l(\vec{\rho}, z^{\text{in}})\right)\mathrm{e}^{\mathrm{i}\,\check{k}_z \Delta z}$ has to be sampled, which may have high spectral frequencies [up till now necessary resolution determined by trial and error [of possibly lower dimensional systems]].
- Propagation behind the focus [expansion] / before the focus [shrinkage] lead to different requirements. A good quantity is the quotient of field size [x percent of the energy of the field] and the whole simulation area.

From the **Rayleigh diffraction formula of first kind** an integral approximation for the **far field** from a field at the focal plane can be derived $[k_0 r \to \infty]$:

$$E_l(\vec{\rho}, z_0 + \Delta z) = -\mathrm{i}\,k_0 n \frac{\Delta z}{r} \frac{\mathrm{e}^{\mathrm{i}\,k_0 n r}}{r} \hat{E}_l\left(k_0 n \frac{\vec{\rho}}{r}, z_0\right) \quad , \quad l \in \{1,2\} \ .$$



One sees the well known result, that the far-field pattern is the spatially Fourier transformed one $\hat{E}_l(k_0 n \frac{\vec{\rho}}{r}, z_0)$ of the one at the focal plane $E_l(\vec{\rho}\,', z_0)$.

As the approximation specifically $R = |\vec{r} - \vec{r}\,'| \approx r - \vec{r}\,' \frac{\vec{r}}{r}$ was used [and that $\frac{1}{r} \gg \frac{1}{r^2}$]. In an other formulation one sees, that $R = \sqrt{[\Delta\rho]^2 + [\Delta z]^2} \approx \Delta z + \frac{[\Delta\rho]^2}{2\Delta z}$.

If one tries to evaluate above approximation with fixed Δz, but is not exactly sure about the position of the focal plane, the only difference in the result will be a phase factor $\mathrm{e}^{\mathrm{i}\,\check{k}_z \Delta z^{\mathrm{in}}}$. [As long, as one is already in the far field!]

Another example of problem-adapted operators is the following one for a **tilted beam** in the reference system. Therefore one finds the effect of the tilt in the k-space [in a linear approximation, propagation direction $\vec{v}$]:

$$a(\vec{\kappa}) = \sqrt{k_0{}^2 n^2 - k_x{}^2 - k_y{}^2}\bigg|_{\vec{\kappa}-\vec{\kappa}_{\vec{v}}=0} \approx \underbrace{k_{z\vec{v}} + \frac{k_{x\vec{v}}{}^2 + k_{y\vec{v}}{}^2}{k_{z\vec{v}}} - \frac{k_{x\vec{v}}{}^2 k_{y\vec{v}}{}^2}{k_{z\vec{v}}{}^3}}_{\text{const.}} + \underbrace{\frac{k_{x\vec{v}}}{k_{z\vec{v}}}\left[\frac{k_{y\vec{v}}{}^2}{k_{z\vec{v}}{}^2} - 1\right]}_{\gamma_{1x}} k_x + \underbrace{\frac{k_{y\vec{v}}}{k_{z\vec{v}}}\left[\frac{k_{x\vec{v}}{}^2}{k_{z\vec{v}}{}^2} - 1\right]}_{\gamma_{1y}} k_y \ .$$

The physical k-values in the direction of the beam are then $\check{\gamma}(\vec{\kappa} - \vec{\kappa}_{\vec{v}}) := \check{k}_z(\vec{\kappa}) - \vec{\gamma}_1 \vec{\kappa}$, with $\vec{\gamma}_1 = \begin{pmatrix}\gamma_{1x}\\ \gamma_{1y}\end{pmatrix}$.

The field at z^{out} can then be written as:

$$V_l(\vec{\rho}, z^{\mathrm{out}}) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} \hat{\tilde{V}}_l(\vec{\kappa}\,', z^{\mathrm{in}})\, \mathrm{e}^{\mathrm{i}\,\check{\gamma}(\vec{\kappa})\Delta z}\, \mathrm{e}^{\mathrm{i}\,\vec{\kappa}\,'[\vec{\rho} + \Delta z \vec{\gamma}_1]}\, \mathrm{d}^2\kappa'\, \mathrm{e}^{\mathrm{i}\,\vec{\kappa}_{\vec{v}}\vec{\rho}}$$

where $\tilde{V}_l(\vec{\rho}, z) = V_l(\vec{\rho}, z)\, \mathrm{e}^{-\mathrm{i}\,\vec{\kappa}_{\vec{v}}\vec{\rho}}$ and $\hat{\tilde{V}}_l(\vec{\kappa}\,', z^{\mathrm{in}}) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} \tilde{V}(\vec{\rho}\,', z^{\mathrm{in}})\, \mathrm{e}^{-\mathrm{i}\,\vec{\kappa}\,'\vec{\rho}\,'}\, \mathrm{d}^2\rho'$.

Thus essentially only 2 Fourier transforms are necessary [which are the numerically most demanding computations].

Alternatively one could rotate the system, propagate the beam and rotate the system back. This however has numerically much higher computational demands [although it is physically the same idea].

Diffraction - everything that is not geometric

Caustic - crossing of geometric rays [→ geometric optics probably erroneous]

Diffraction limited beam - purely spherical phase in far-field [here geometric approximation only valid in far field]