electric field $\tilde{E}$ free space permittivity ε_0
susceptibility χ polarization $\tilde{P} = \varepsilon_0 \chi \tilde{E}$
refractive index $n = \sqrt{\varepsilon_r} = \sqrt{1+\chi}$
electric displacement field $\tilde{D} = \varepsilon_0 \varepsilon_r \tilde{E} = \varepsilon_0 \tilde{E} + \tilde{P}$
electric charge density $\tilde{\rho}$ electric current density $\tilde{\jmath}$
magnetic field $\tilde{B}$ free space permeability μ_0
magnetization field $\tilde{M}$
magnetizing field $\tilde{H} = \frac{1}{\mu_0}\tilde{B} - \tilde{M}$ free space velocity of light $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

Maxwell's Equations (macroscopic)

Faraday's law	$\nabla \times \tilde{E}(r,t) + \frac{\partial}{\partial t}\tilde{B}(\vec{r},t)$	=	0
Ampère's law	$\nabla \times \tilde{H}(r,t) - \frac{\partial}{\partial t}\tilde{D}(r,t)$	=	$\tilde{\jmath}(r,t)$
Gauss's law	$\nabla \tilde{D}(r,t)$	=	$\tilde{\rho}(r,t)$
Gauss's law	$\nabla \tilde{B}(r,t)$	=	0

Pointing: $\vec{S} = \vec{E} \times \vec{H}$
electric energy in a medium: $w_e = \frac{1}{2}\vec{E}\vec{D}$

In an anisotropic crystal with $\vec{k} = \frac{\omega n}{c}\vec{s}$, $|\vec{s}| = 1$ it follows $\vec{D} = \overbrace{\frac{n^2}{c^2 \mu_0^2}}^{\tilde{n}^2}[\vec{E} - \vec{s}[\vec{s}\vec{E}]]$.

In the principle axis system [HAS - Hauptachsensystem] of an anisotropic crystal $\hat{\varepsilon} = \varepsilon_0 \begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix}$.

Then it follows Fresnel's equation [in HAS]: $$\frac{s_x^2}{\frac{1}{\tilde{n}^2} - \frac{\varepsilon_0}{\varepsilon_x}} + \frac{s_y^2}{\frac{1}{\tilde{n}^2} - \frac{\varepsilon_0}{\varepsilon_y}} + \frac{s_z^2}{\frac{1}{\tilde{n}^2} - \frac{\varepsilon_0}{\varepsilon_z}} = 0 \ .$$

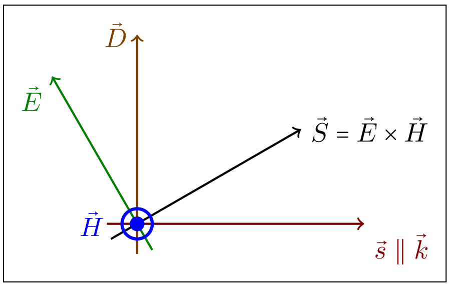
From w_e it follows in the principle axis system $[x := \frac{D_x}{\sqrt{2w_e}}, n_x = \sqrt{\varepsilon_x}]$ the index ellipsoid: $\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1$.

The axis orthogonal to the plane, which's intersection with the ellipsoide is a circle, is called **optical axes**.

[In general exist 2, in uniaxial crystals only 1.]

It is $\vec{s} = \frac{c}{\omega n}\vec{k}$, $\vec{s} \times \vec{H} = \frac{c}{n}\vec{D}$, $\vec{s} \times \vec{E} = -\frac{c}{n}\mu_0 \vec{H}$.

The plane spanned by the $\vec{k}$-vector of the incident beam and the optical axis of the chrystal is called „**Hauptschnitt**". The $\vec{E}$-field component parallel to this plane is called „**extraordinary**", the polarization orthogonal to it „**ordinary**".



The double refracted $\vec{E}$-field vector is always orthogonal to the indexellipsoide.

Nonlinear optical effects:
approach for the induced polarization in the media $P(t) = \varepsilon_0[\chi^{(1)}E(t) + \chi^{(2)}E^2(t) + \chi^{(3)}E^3(t) + \ldots]$.

By using the quadratic effect and with two incident frequencies ω_1, ω_2 the following are excited in the medium: $2\omega_1$, $2\omega_2$, $\omega_1 + \omega_2$ [SFG - sum frequency generation], $\omega_1 - \omega_2$ [DFG - difference frequency generation] and 0 [rectification]. At least principally - in reality phase-matching conditions have to be considered additionally.

For the third order process 44 different frequencies can be excited.

parametric process: only virtual energy-levels involved [statess exist shorter than allowed by Heisenberg].

non-parametric processes: energy-storing and loss in medium possible.

Taking into account a timely response function of the medium, in the linear case it yealds:

$P^{(1)}(\vec{r},t) = \varepsilon_0 \chi^{(1)}(t) * E(\vec{r},t)$ in time-domain and $P^{(1)}(\vec{r},f) = \varepsilon_0 \chi^{(1)}(f) E(\vec{r},f)$ in frequency domain.

[convolution $a(t) * b(t) = \int_{\mathbb{R}} a(t-\tau) b(\tau)\,\mathrm{d}\tau$]

For $P^{(n)}$ there are n convolutions in time-domain and analogously $n-1$ in frequency-domain.

refractive index $n^2 = \varepsilon = 1 + \chi$

$$\cos(x)\cos(y) = \tfrac{1}{2}[\cos(x+y) + \cos(x-y)] \qquad \sin(x)\sin(y) = \tfrac{1}{2}[\cos(x-y) - \cos(x+y)]$$

$$\cos^3(x) = \tfrac{1}{4}\cos(3x) + \tfrac{3}{4}\cos(x) \qquad a\cos(\omega t) + b\cos(\omega t + \theta) = \sqrt{a^2 + 2ab\cos(\theta) + b^2}\cos\left(\omega t + \arctan\left(\frac{b\sin(\theta)}{a + b\cos(\theta)}\right)\right)$$

Anharmonic oscillator model [1D]:

approach is to make the driving force non-linear: $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{q}{m}E(t) - [ax^2 + bx^3 + \ldots]$.
In order to have an emitted radiation, the oscillator has oscillate and can't be damped to strongly $[0 \leq \gamma < 2\omega_0]$.

Number of atoms N, charge q and displacement x, then $P = \overbrace{Nq}^{=\eta}\, x$ and thus

$$\ddot{P} + \gamma\dot{P} + \omega_0^2 P = \frac{Nq^2}{m}E(t) - [a\eta P^{(2)} + b\eta^2 P^{(3)} + \ldots] \quad .$$

The formal solution is $P(t) = s(t) * Q(t)$ with $s(t) = \frac{Nq^2}{m}E(t) - [a\eta P^{(2)} + b\eta^2 P^{(3)} + \ldots]$ and $Q(t) = \int_{\mathbb{R}} \frac{1}{\omega_0^2 - \mathrm{i}\gamma\omega - \omega^2}\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}f$.
This can ba approximated by perturbation theory for small perturbations or numerically [iteratively!].
In first order it again follows $P^{(1)}(t) = \frac{Nq^2}{m}Q(t) * E(t)$ or $P^{(1)}(f) = \frac{N^2 q}{m}Q(f)E(f)$ respectively.

Intrinsic permutation symmetry: $\chi_{ijk}(f_3; -f_1, -f_2) = \chi^*_{ijk}(-f_3; f_2, f_1)$
Full permutation symmetry: $\chi_{ijk}(-f_3; f_1, f_2) = \chi_{jik}(f_1; -f_3, f_2)$
Kleinmann's symmetry [far off resonance]: Media are lossless and χ is frequency independent.
Nonlinear coefficient: $d_{ijk} = \frac{1}{2}\chi^{(2)}_{ijk}$ [used if Kleinmann's symmetry holds -> frequency independent]

contracted indices:

l	1	2	3	4	5	6
j	1	2	3	2	1	1
k	1	2	3	3	3	2

nonlinear wave equation [assuming $\vec{\nabla}\tilde{E} \approx 0$]: $-\vec{\nabla}^2\tilde{\vec{E}} + \frac{1}{\varepsilon_0 c^2}\frac{\partial^2 \tilde{\vec{D}}^{(1)}}{\partial t^2} = -\frac{1}{\varepsilon_0 c^2}\frac{\partial^2 \tilde{\vec{P}}^{NL}}{\partial t^2}$, with $\tilde{\vec{D}}^{(1)} = \varepsilon_0 \tilde{\vec{E}} + \tilde{\vec{P}}^{(1)}$.

Coupled wave equations for sum frequency generation [SFG; lossless media, slowly varying amplitude approximation, $\tilde{E}_j = A_j(z)\,\mathrm{e}^{\mathrm{i}[k_j z - \omega_j t]} + c.c.$, A_1 and A_2 incident, A_3 produced, $\Delta k = k_1 + k_2 - k_3$]:

$$\frac{\partial A_1}{\partial z} = \mathrm{i}\,\frac{2d_{\text{eff}}\omega_1^2}{k_1 c^2}A_3 A_2^*\,\mathrm{e}^{-\mathrm{i}\Delta k z} \qquad \frac{\partial A_2}{\partial z} = \mathrm{i}\,\frac{2d_{\text{eff}}\omega_2^2}{k_2 c^2}A_3 A_1^*\,\mathrm{e}^{-\mathrm{i}\Delta k z} \qquad \frac{\partial A_3}{\partial z} = \mathrm{i}\,\frac{2d_{\text{eff}}\omega_3^2}{k_3 c^2}A_1 A_2\,\mathrm{e}^{\mathrm{i}\Delta k z}$$

Manley-Rowe-Relations for SFG $[I_i = \frac{1}{2}cn_i\varepsilon_0|A_i|^2]$: $\frac{\mathrm{d}}{\mathrm{d}z}\frac{I_1}{\omega_1} = \frac{\mathrm{d}}{\mathrm{d}z}\frac{I_2}{\omega_2} = \frac{\mathrm{d}}{\mathrm{d}z}\frac{I_3}{\omega_3}$.

It holds $\frac{\mathrm{d}}{\mathrm{d}z}I_{\text{ges}} = \frac{\mathrm{d}}{\mathrm{d}z}[I_1 + I_2 + I_3] = 0$.

For second harmonic generation [SHG] the same relations as for SFG hold, one only has to convert $1, 2 \to 1'$, $3 \to 2'$:

$$\frac{\partial A_{1'}}{\partial z} = \mathrm{i}\,\frac{2d_{\text{eff}}\omega_{1'}^2}{k_{1'}c^2}A_{2'}A_{1'}^*\,\mathrm{e}^{-\mathrm{i}\Delta k z} \qquad \frac{\partial A_{2'}}{\partial z} = \mathrm{i}\,\frac{2d_{\text{eff}}\omega_{2'}^2}{k_{2'}c^2}A_{1'}^2\,\mathrm{e}^{\mathrm{i}\Delta k z}$$

From the third order effect it follows a change in the refractive index: $n = n_0 + \frac{3}{4}\frac{1}{n_0^2\varepsilon_0 c}\chi^{(3)} I$.

$[I = 2n_0\varepsilon_0 c|E(\omega)|^2$, $\tilde{E}(t) = E(\omega)\,\mathrm{e}^{-\mathrm{i}\omega t} + c.c.$, real field $\mathcal{E} = \frac{1}{2}\tilde{E}(t)]$

In an uniaxial crystal [positive $n_e > n_o$; negative $n_e < n_o$]: $n(\theta) = \frac{n_e n_o}{\sqrt{n_o^2\sin^2(\theta) + n_e^2\cos^2(\theta)}}$.

Solitons: The chirp due to second order diffraction cancels the chirp due to self phase modulation! Solution:
$\Psi = \phi_0\,\mathrm{sech}\left(\frac{x - u_e t}{L_e}\right)\mathrm{e}^{\mathrm{i}[\kappa x - \mu t]}$, with $L_e = \frac{1}{\phi_0}\sqrt{\frac{2P}{Q}}$, $\kappa = \frac{u_e}{2P}$, $\mu = \frac{u_e u_p}{2P}$, group velocity u_e, phase velocity u_p, $u_e \neq u_p$, following from the NLS $\mathrm{i}\frac{\partial\Psi}{\partial t} = \left[-P\frac{\partial^2}{\partial x^2} - Q|\Psi|^2\right]\Psi$.