Zwangsbedingungen: $\tilde{g}_\alpha(q_A,t) = 0 = g_\alpha(x_n^i(q_A,t),t)$ Lagrange I: $m_n \ddot{x}_n^i = K_n^i + \sum_\alpha \lambda_\alpha \frac{\partial g_\alpha}{\partial x_n^i} \quad \forall i,n.$

$\Rightarrow 0 = \frac{\partial g_\alpha}{\partial q_A} = \sum_{i,n} \frac{\partial g_\alpha}{\partial x_n^i}\frac{\partial x_n^i}{\partial q_A}$ q_A - **generalisierte Koordinaten** $[A = 1,\dots,N_F]$,

N_Z - Anzahl Zwangsbedingungen
N_F - Anzahl Freiheitsgrade
N - Anzahl Punkte

$\sum_{i,n} \frac{\partial x_n^i}{\partial q_A} \cdot [\text{Lagr. I}]_n^i$:

$$\sum_{i,n} m_n \ddot{x}_n^i \frac{\partial x_n^i}{\partial q_A} = \sum_{i,n} K_n^i \frac{\partial x_n^i}{\partial q_A} + \lambda_\alpha \underbrace{\sum_{i,n} \frac{\partial g_\alpha}{\partial x_n^i}\frac{\partial x_n^i}{\partial q_A}}_{=0}$$

„$\sum_{i,n}$“ $\hat{=} \sum_{i=1}^{3}\sum_{n=1}^{N}$

„$\sum_B$“ $\hat{=} \sum_{B=1}^{N_F}$

„$\sum_\alpha$“ $\hat{=} \sum_{\alpha=1}^{N_Z}$

$$\Rightarrow \sum_{i,n} m_n \ddot{x}_n^i \frac{\partial x_n^i}{\partial q_A} - \sum_{i,n} K_n^i \frac{\partial x_n^i}{\partial q_A} = 0$$

Es ist: $\dot{x}_n^i = \frac{\mathrm{d}x_n^i}{\mathrm{d}t} = \sum_B \frac{\partial x_n^i}{\partial q_B}\dot{q}_B + \frac{\partial x_n^i}{\partial t} \quad \Rightarrow \quad \frac{\partial \dot{x}_n^i}{\partial q_A} = \sum_B \frac{\partial^2 x_n^i}{\partial q_A \partial q_B}\dot{q}_B + \frac{\partial^2 x_n^i}{\partial q_A \partial t} \quad , \quad \frac{\partial \dot{x}_n^i}{\partial \dot{q}_A} = \frac{\partial x_n^i}{\partial q_A}$

Mit der **kinetischen Energie** $T = \sum_{i,n} \frac{1}{2} m_n [\dot{x}_n^i]^2 = \sum_{i,n} \frac{1}{2} m_n \left[\frac{\partial x_n^i}{\partial q_B}\dot{q}_B + \frac{\partial x_n^i}{\partial t}\right]^2$ folgt:

$$\begin{aligned}
\frac{\partial T}{\partial q_A} &= \sum_{i,n} m_n \dot{x}_n^i \frac{\partial \dot{x}_n^i}{\partial q_A} = \sum_{i,n} m_n \dot{x}_n^i \left[\sum_B \frac{\partial^2 x_n^i}{\partial q_A \partial q_B}\dot{q}_B + \frac{\partial^2 x_n^i}{\partial q_A \partial t}\right], \\
\frac{\partial T}{\partial \dot{q}_A} &= \sum_{i,n} m_n \dot{x}_n^i \frac{\partial \dot{x}_n^i}{\partial \dot{q}_A} = \sum_{i,n} m_n \dot{x}_n^i \frac{\partial x_n^i}{\partial q_A}, \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_A} &= \sum_{i,n} m_n \ddot{x}_n^i \frac{\partial x_n^i}{\partial q_A} + \sum_{i,n} m_n \dot{x}_n^i \frac{\partial}{\partial q_A}\frac{\mathrm{d}x_n^i}{\mathrm{d}t} = \underbrace{\sum_{i,n} m_n \ddot{x}_n^i \frac{\partial x_n^i}{\partial q_A}}_{Q_A} + \underbrace{\sum_{i,n} m_n \dot{x}_n^i \left[\sum_B \frac{\partial^2 x_n^i}{\partial q_A \partial q_B}\dot{q}_B + \frac{\partial^2 x_n^i}{\partial q_A \partial t}\right]}_{=\frac{\partial T}{\partial q_A}}.
\end{aligned}$$

Mit den **generalisierten Kräften** $Q_A = \sum_{i,n} K_n^i \frac{\partial x_n^i}{\partial q_A}$ schreibt sich dies als:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_A} - \frac{\partial T}{\partial q_A} = Q_A \,.$$

Weisen die Kräfte ein **Potential** auf $[K_n^i = -\frac{\partial U}{\partial x_n^i}$; mit $U = U(x_n^i,t)$, $\frac{\partial U}{\partial \dot{q}_A} = 0]$, so ist:

$$\frac{\partial U}{\partial q_A} = \sum_{i,n} \frac{\partial U}{\partial x_n^i}\frac{\partial x_n^i}{\partial q_A} \qquad \Rightarrow \; Q_A = \sum_{i,n} K_n^i \frac{\partial x_n^i}{\partial q_A} = -\sum_{i,n} \frac{\partial U}{\partial x_n^i}\frac{\partial x_n^i}{\partial q_A} = -\frac{\partial U}{\partial q_A}$$

$$\Longrightarrow \; \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_A} - \frac{\partial T}{\partial q_A} = -\frac{\partial U}{\partial q_A} + [\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial U}{\partial \dot{q}_A}}_{=0}].$$

Mit der Lagrange Funktion $L(q_A,\dot{q}_A,t) = T(q_A,\dot{q}_A,t) - U(q_A,t)$ formuliert sich so der **Lagrange II** - Formalismus:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_A} - \frac{\partial L}{\partial q_A} = 0$$

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \sum_B \frac{\partial L}{\partial q_B}\dot{q}_B + \sum_B \frac{\partial L}{\partial \dot{q}_B}\ddot{q}_B + \frac{\partial L}{\partial t} = \sum_B \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}_B}\right]\dot{q}_B + \sum_B \frac{\partial L}{\partial \dot{q}_B}\ddot{q}_B + \frac{\partial L}{\partial t} = \sum_B \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}_B}\dot{q}_B\right] + \frac{\partial L}{\partial t}$$

Dies ist die **Energiebilanzgleichung** im Lagrange-Formalismus: $\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_B \frac{\partial L}{\partial \dot{q}_B}\dot{q}_B - L\right] = -\frac{\partial L}{\partial t}$.

Das **Wirkungsfunktional** S ist wie folgt definiert: $S(q_A) = \int_{t_1}^{t_2} L(q_A, \dot{q}_A, t)\,\mathrm{d}t$.

Die **generalisierten Impulse** p_A sind: $p_A = \dfrac{\partial L}{\partial \dot{q}_A}$. (q_A, p_A) heißen **kanonisch konjugiert**.

Die **Hamilton-Funktion** H ist: $H(q_A, p_A, t) = \sum_B \dot{q}_B\, p_B - L$ [vgl. Energiebilanz]

Die **kanonischen Gleichungen**:

$$\begin{aligned}
\frac{\partial H}{\partial q_A} &= \cancel{\sum_B \frac{\partial \dot{q}_B}{\partial q_A}} p_B - \frac{\partial L}{\partial q_A} - \cancel{\sum_B \frac{\partial L}{\partial \dot{q}_B} \frac{\partial \dot{q}_B}{\partial q_A}} = -\frac{\partial L}{\partial q_A} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_A} = -\dot{p}_A \\
\frac{\partial H}{\partial p_A} &= \cancel{\sum_B \frac{\partial \dot{q}_B}{\partial p_A}} p_B + \dot{q}_A - \cancel{\sum_B \frac{\partial L}{\partial \dot{q}_B} \frac{\partial \dot{q}_B}{\partial p_A}} = \dot{q}_A \\
\frac{\partial H}{\partial t} &= \cancel{\sum_B \frac{\partial \dot{q}_B}{\partial t}} p_B - \cancel{\sum \frac{\partial L}{\partial \dot{q}_B} \frac{\partial \dot{q}_B}{\partial t}} - \frac{\partial L}{\partial t} = -\frac{\partial L}{\partial t}
\end{aligned}$$

Zyklische Koordinaten: $\dfrac{\partial H}{\partial q_A} = -\dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dfrac{\partial L}{\partial \dot{q}_A}\right] = 0 \quad \Rightarrow \quad p_A = const.$ [zeitlich].

Poisson-Klammern: $\{F, G\} = \sum_B \left[\dfrac{\partial F}{\partial q_B}\dfrac{\partial G}{\partial p_B} - \dfrac{\partial F}{\partial p_B}\dfrac{\partial G}{\partial q_B}\right] = -\{G, F\}$

$$\Rightarrow \frac{\mathrm{d}F}{\mathrm{d}t} = \sum_B \frac{\partial F}{\partial q_B}\dot{q}_B + \sum_B \frac{\partial F}{\partial p_B}\dot{p}_B + \frac{\partial F}{\partial t} = \sum_B \frac{\partial F}{\partial q_B}\frac{\partial H}{\partial p_B} - \sum_B \frac{\partial F}{\partial p_B}\frac{\partial H}{\partial q_B} + \frac{\partial F}{\partial t} = \{F, H\} + \frac{\partial F}{\partial t} .$$

Kanonische Transformationen $[Q_A = Q_A(q_B, p_B, t),\ P_A = P_A(q_B, p_B, t),\ H' = H'(Q_A, P_A, t)]$ lassen die kanonischen Gleichungen invariant. $[\dfrac{\partial H}{\partial P_A} = \dot{Q}_A,\ \dfrac{\partial H}{\partial Q_A} = -\dot{P}_A]$

Auch die Poisson-Klammern sind invariant gegenüber diesen Transformationen.

$$[\{F, G\} = \sum_B \left[\frac{\partial F}{\partial q_B}\frac{\partial G}{\partial p_B} - \frac{\partial F}{\partial p_B}\frac{\partial G}{\partial q_B}\right] = \sum_B \left[\frac{\partial F}{\partial Q_B}\frac{\partial G}{\partial P_B} - \frac{\partial F}{\partial P_B}\frac{\partial G}{\partial Q_B}\right]]$$

Die Transformation erfolgt über **erzeugende Funktionen**, derer es 4 Klassen gibt:

				Legendre-Transformation
$R_1 = R_1(q_A, Q_A, t)$	$\dfrac{\partial R_1}{\partial q_A} = p_A$	$\dfrac{\partial R_1}{\partial Q_A} = -P_A$	$\dfrac{\partial R_1}{\partial t} = H' - H$	
$R_2 = R_2(q_A, P_A, t)$	$\dfrac{\partial R_2}{\partial q_A} = p_A$	$\dfrac{\partial R_2}{\partial P_A} = Q_A$	$\dfrac{\partial R_2}{\partial t} = H' - H$	$R_2 = R_1 + \sum_B Q_B P_B$
$R_3 = R_3(p_A, Q_A, t)$	$\dfrac{\partial R_3}{\partial p_A} = -q_A$	$\dfrac{\partial R_3}{\partial Q_A} = -P_A$	$\dfrac{\partial R_3}{\partial t} = H' - H$	$R_3 = R_1 - \sum_B q_B p_B$
$R_4 = R_4(p_A, P_A, t)$	$\dfrac{\partial R_4}{\partial p_A} = -q_A$	$\dfrac{\partial R_4}{\partial P_A} = Q_A$	$\dfrac{\partial R_4}{\partial t} = H' - H$	$R_4 = R_1 - \sum_B q_B p_B + \sum_B Q_B P_B$

Fordert man nun $H' \overset{!}{=} 0$, so gilt für R_2: $p_A = \frac{\partial R_2}{\partial q_A}$, $H(q_A, \frac{\partial R_2}{\partial q_A}, t) + \frac{\partial R_2}{\partial t} = 0$, $P_A = const.\ \forall A$.

$$\Rightarrow \frac{\mathrm{d}R_2}{\mathrm{d}t} = \sum_B \frac{\partial R_2}{\partial q_B}\dot{q}_B + \sum_B \frac{\partial R_2}{\partial P_B}\dot{P}_B + \frac{\partial R_2}{\partial t} = \sum_B p_B \dot{q}_B + \sum_B Q_B \underbrace{\dot{P}_B}_{=0} + \frac{\partial R_2}{\partial t} = \sum_B p_B \dot{q}_B - H(q_A, \frac{\partial R_2}{\partial q_A}, t) = L(q_A, \frac{\partial R_2}{\partial q_A}, t)$$

$\Rightarrow R_2 = \int L\,\mathrm{d}t = S$ entspricht dem Wirkungsfunktional!

Dies ist die **Hamilton-Jacobi-Gleichung**:

$$\frac{\partial S}{\partial t} + H(q_A, \frac{\partial S}{\partial q_A}, t) = 0$$