Formelsammlung zur Vektoranalysis

1 Vektor-Differentialoperatoren in krummlinigen Orthogonalkoordinaten

Linienelement: $ds^2 = {g_1}^2 {dx_1}^2 + {g_2}^2 {dx_2}^2 + {g_3}^2 {dx_3}^2$

Beispiele:

	x_1	x_2	x_3	g_1	g_2	g_3
kartes.	x	y	z	1	1	1
Zylinder	r	φ	z	1	r	1
Kugel	r	ϑ	φ	1	r	$r\sin(\vartheta)$

Volumenelement: $dV = g_1 g_2 g_3 dx_1 dx_2 dx_3$

Gradient: $\text{grad}(U) = \left(\frac{1}{g_1}\frac{\partial U}{\partial x_1}, \frac{1}{g_2}\frac{\partial U}{\partial x_2}, \frac{1}{g_3}\frac{\partial U}{\partial x_3}\right)$

Divergenz: $\text{div}(\vec{A}) = \frac{1}{g_1 g_2 g_3}\left[\frac{\partial}{\partial x_1}(A_1 g_2 g_3) + \frac{\partial}{\partial x_2}(g_1 A_2 g_3) + \frac{\partial}{\partial x_3}(g_1 g_2 A_3)\right]$

Rotation: $\text{rot}(\vec{A}) = \frac{1}{g_1 g_2 g_3}\begin{vmatrix} g_1\vec{e}_1 & g_2\vec{e}_2 & g_3\vec{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ g_1 A_1 & g_2 A_2 & g_3 A_3 \end{vmatrix}$

Laplace-Operator: $\triangle U = \text{div}(\text{grad}(U)) = \frac{1}{g_1 g_2 g_3}\left[\frac{\partial}{\partial x_1}\left(\frac{g_2 g_3}{g_1}\frac{\partial U}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{g_1 g_3}{g_2}\frac{\partial U}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{g_1 g_2}{g_3}\frac{\partial U}{\partial x_3}\right)\right]$

2 Rechenregeln

$\text{rot}(\text{grad}(U)) = 0$

$\text{div}(\text{rot}(\vec{A})) = 0$

$\text{rot}(\text{rot}(\vec{A})) = \text{grad}(\text{div}(\vec{A})) - \triangle\vec{A}$ (kartesische Koordinaten)

$\text{div}(\vec{A}\times\vec{B}) = \vec{B}\,\text{rot}(\vec{A}) - \vec{A}\,\text{rot}(\vec{B})$

$\text{grad}(UV) = U\,\text{grad}(V) + V\,\text{grad}(U)$

$\text{rot}(\lambda\vec{A}) = \lambda\,\text{rot}(\vec{A}) + (\text{grad}(\lambda))\times\vec{A}$

$\text{div}(\lambda\vec{A}) = \lambda\,\text{div}(\vec{A}) + \vec{A}\,\text{grad}(\lambda)$

$\text{rot}(\vec{A}\times\vec{B}) = (\vec{B}\text{grad})\vec{A} - (\vec{A}\text{grad})\vec{B} + \vec{A}\,\text{div}(\vec{B}) - \vec{B}\,\text{div}(\vec{A})$

$\text{grad}(\vec{A}\vec{B}) = (\vec{B}\text{grad})\vec{A} + (\vec{A}\text{grad})\vec{B} + \vec{A}\times\text{rot}(\vec{B}) + \vec{B}\times\text{rot}(\vec{A})$

3 Vektor-Differentialoperatoren in Zylinder- und Kugelkoordinaten

a) Zylinderkoordinaten

Linienelement: $ds^2 = dr^2 + r^2\, d\varphi^2 + dz^2$

Volumenelement: $dV = r\, dr\, d\varphi\, dz$

Gradient: $\text{grad}(U) = \frac{\partial U}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial U}{\partial \varphi}\vec{e}_\varphi + \frac{\partial U}{\partial z}\vec{e}_z$

Divergenz: $\text{div}(\vec{A}) = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$

Rotation: $\text{rot}(\vec{A}) = \left[\frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right]\vec{e}_r + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\vec{e}_\varphi + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\varphi) - \frac{\partial A_r}{\partial \varphi}\right]\vec{e}_z$

Laplace-Operator: $\triangle U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \varphi^2} + \frac{\partial^2 U}{\partial z^2}$

b) Kugelkoordinaten

Linienelement: $ds^2 = dr^2 + r^2\, d\vartheta^2 + r^2\, \sin^2(\vartheta)\, d\varphi^2$

Volumenelement: $dV = r^2\, \sin(\vartheta)\, dr\, d\vartheta\, d\varphi$

Gradient: $\text{grad}(U) = \frac{\partial U}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial U}{\partial \vartheta}\vec{e}_\vartheta + \frac{1}{r\, \sin(\vartheta)}\frac{\partial U}{\partial \varphi}\vec{e}_\varphi$

Divergenz: $\text{div}(\vec{A}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{a}{r\, \sin(\vartheta)}\left[\frac{\partial}{\partial \vartheta}(\sin(\vartheta)\, A_\vartheta) + \frac{\partial A_\varphi}{\partial \varphi}\right]$

Rotation: $\text{rot}(\vec{A}) = \frac{1}{r\, \sin(\vartheta)}\left[\frac{\partial}{\partial \vartheta}(\sin(\vartheta)\, A_\varphi) - \frac{\partial A_\vartheta}{\partial \varphi}\right]\vec{e}_r + \frac{1}{r}\left[\frac{1}{r\, \sin(\vartheta)}\frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r}(r\, A_\varphi)\right]\vec{e}_\vartheta + \frac{1}{r}\left[\frac{\partial}{\partial r}(r\, A_\vartheta) - \frac{\partial A_r}{\partial \vartheta}\right]\vec{e}_\varphi$

Laplace-Operator: $\triangle U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\, \sin(\vartheta)}\frac{\partial}{\partial \vartheta}\left(\sin(\vartheta)\frac{\partial U}{\partial \vartheta}\right) + \frac{1}{r^2\, \sin^2(\vartheta)}\frac{\partial^2 U}{\partial \varphi^2}$